SYMMETRIC 2-STEP 4-POINT HYBRID METHOD FOR THE SOLUTION OF GENERAL THIRD ORDER DIFFERENTIAL EQUATIONS

S. J. Kayode and F.O. Obarhua*
Department of Mathematical Sciences, Federal University of Technology, P. M. B. 704, Akure, Nigeria.

Abstract: This research considers a symmetric hybrid continuous linear multistep method for the solution of general third order ordinary differential equations. The method is generated by interpolation and collocation approach using a combination of power series and exponential function as basis function. The approximate basis function is interpolated at both grid and off-grid points but the collocation of the differential function is only at the grid points. The derived method was found to be symmetric, consistent, zero stable and of order six with low error constant. Accuracy of the method was confirmed by implementing the method on linear and non-linear test problems. The results show better performance over known existing methods solved with the same third order problems.

Keywords: Symmetric, Hybrid method, Power series and Exponential function, Continuous Predictor-Corrector method.

AMS 2010 Subject Classification: 65D05; 65L05; 65L06.

***Corresponding Author's e-mail address:** obycoach@gmail.com

1. Introduction

The study of thin film flow in fluid dynamics, electromagnetic waves, gravity driven flow, transport and disposition of chemicals through the body, immune-assay chemistry for developing new blood test have been the applicable mathematical tasks which are drastically changed in recent years. The modeling of these physical and biological problems give rise to different forms of ordinary differential equations (odes) of different orders and forms which is generally represented in the form

$$y^{(\mu)} = f\left(t, y, y', ..., y^{(\mu-1)}\right), \; y(t_0) = y_0, \; y'(t_0) = y_1, ..., y^{(\mu-1)}(t_0) = y_{\mu-1} \qquad (1)$$

where μ is the order of the problem, $y \in C^{\mu}[a,b]$ and f is a continuous function of t, y, and derivatives of y.

In this paper, we considered the solution of initial value problems when $\mu = 3$ in (1) of the form

$$y''' = f\left(t, \; y \; y', \; y''\right), \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y''(t_0) = y_2. \qquad (2)$$

where $t, y, f \in \mathbb{R}$.

The numerical and theoretical studies of equation (2) have appeared in literature severally. The direct approach for solving this type of ordinary differential equations have been studied and appeared in different literature, “[1], [2], [3], [4], [5], [6]”. This direct approach has demonstrated advantages over the popular approach (reduction to system of first order approach) in terms of speed and accuracy [7]. Many authors have focused on direct solution of general second order ivps of odes of the form

$$y'' = f\left(t, y, y'\right). \tag{3}$$

Majid, *et al.* [8] proposed two point four step direct implicit block method for the solution of second order systems of ordinary differential equations (ODEs), using variable step size. The method estimated the solutions of initial value problems at two points simultaneously by using four backward steps but with lower order of accuracies. Akinfenwa [9] presented ninth order hybrid block integrator for solving second order ordinary differential equations. In the paper, the proposed block integrator discretizes the problem using the main and the additional methods to generate systems of equations. The resulting system was solved simultaneously in a block-by-block fashion but the order of accuracies is low compared to the order of the method. Kayode and Adeyeye [3] came up with direct implementation of predictor-corrector methods. The authors emphasized the need to develop the same order of accuracy of the main predictors and that of the correctors to ensure good accuracy of the method. The order of accuracies in these works improved significantly compare to the existing methods with lower order of their main predictors. Ogunware *et al.* [10]. studied hybrid and non-hybrid implicit schemes for solving equation (2) using block method as predictors.

Attempts have also been made by these scholars, “[3], [11], [12], [13], [14]” to mention a few. Olabode [13], proposed a 5-step block scheme for the solution of special type of (1). The order of accuracy in [13] improves more than that of [12]. Awoyemi, *et al.* [5], developed a four-point implicit method for the numerical integration of third order ODEs using power series polynomial function in [15]. Kuboye and Omar [6] proposed numerical solution of third order ordinary differential equations using a seven-step block method to improve on Awoyemi *et al.* [5] and Olabode [13] which are of lower order of accuracy. Odekunle *et al.* [16] later investigated a new block integrator for the solution of ivps of first order odes using the combination of power series and exponential function as the basis function. Furthermore, symmetric hybrid linear multistep method of order six having two off-step points for the solution of (2) directly was presented in [17].

To improve on the study in [17], a symmetric of two-step four-point hybrid method for the solution of third order initial value problems of ordinary differential equations directly is therefore proposed. Combination of power series and exponential function was used as the approximate basis function in generating the interpolation and collocation equations for the development of continuous hybrid linear multistep method for the direct solution of problem (2).

2. Derivation of the Method

This research work considers the derivation of 2-step 4-point hybrid method for the solution of general third order initial value problems of ordinary differential equations. The approach is to solve (2) directly without reducing it to a system of first order differential equation. A combination of power series and exponential function is used as the basis function for (2). The approximate solution (2) and the resulting differential systems are respectively given as

$$y(t) = \sum_{j=0}^{r+s-1} \lambda_j t^j + \lambda_{r+s} \sum_{j=0}^{r+s} \frac{\alpha^j t^j}{j!}. \tag{4}$$

where λ_j is an unknown parameters, $t \in [a, b]$, the solution interval, α is a free parameter while r and s are the number of interpolation and collocation points respectively.

The third derivative of (4) as compared with (2) gives

$$f\left(t,\ y,\ y'\ y''\right) = \sum_{j=3}^{r+s-1} \left(j(j-1)(j-2)\lambda_j t^{j-3} \right) + \lambda_{r+s} \sum_{j=3}^{r+s-1} \frac{\alpha^j t^{j-3}}{(j-3)!}. \tag{5}$$

Collocating (5) at only the grid points, $t_{n+j},\quad j = 0(1)2,$ and interpolating (4) at both grid and off-grid points, $t_{n+j},\quad j = 0\left(\frac{1}{3}\right)2$, leads to the following system of equations written in matrix form (6).

$$At = b \tag{6}$$

where $t = [\lambda_0\ .\ .\ .\ \lambda_8]^T;\ b = [y_n\ .\ .\ .\ f_{n+2}]^T$

$$A = \begin{bmatrix} 1 & t_n & t_n^2 & . . . & t_n^k & \delta_1 \\ 1 & t_{n+\frac{1}{3}} & t^2_{n+\frac{1}{3}} & . . . & t^k_{n+\frac{1}{3}} & \delta_2 \\ 1 & t_{n+\frac{2}{3}} & t^2_{n+\frac{2}{3}} & . . . & t^k_{n+\frac{2}{3}} & \delta_3 \\ 1 & t_{n+1} & t^2_{n+1} & . . . & t^k_{n+1} & \delta_4 \\ 1 & t_{n+\frac{4}{3}} & t^2_{n+\frac{4}{3}} & . . . & t^k_{n+\frac{4}{3}} & \delta_5 \\ 1 & t_{n+\frac{5}{3}} & t^2_{n+\frac{5}{3}} & . . . & t^k_{n+\frac{5}{3}} & \delta_6 \\ 0 & 0 & 0 & 6 & 24t_n\ .\ .\ .\ \psi t_n^4 & \eta_1 \\ 0 & 0 & 0 & 6 & 24t_{n+1} .\ .\ .\ \psi t^4_{n+1} & \eta_2 \\ 0 & 0 & 0 & 6 & 24t_{n+2} .\ .\ .\ \psi t^4_{n+2} & \eta_3 \end{bmatrix}$$

where

$$\delta_1 = [1+\alpha t_n + \frac{\alpha^2 t_n^2}{2!} + \ldots + \frac{\alpha^8 t_n^8}{8!}]$$
$$\delta_2 = [1+\alpha t_{n+\frac{1}{3}} + \frac{\alpha^2 t_{n+\frac{1}{3}}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+\frac{1}{3}}^8}{8!}]$$
$$\delta_3 = [1+\alpha t_{n+\frac{2}{3}} + \frac{\alpha^2 t_{n+\frac{2}{3}}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+\frac{2}{3}}^8}{8!}]$$
$$\delta_4 = [1+\alpha t_{n+1} + \frac{\alpha^2 t_{n+1}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+1}^8}{8!}]$$
$$\delta_5 = [1+\alpha t_{n+\frac{4}{3}} + \frac{\alpha^2 t_{n+\frac{4}{3}}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+\frac{4}{3}}^8}{8!}]$$
$$\delta_6 = [1+\alpha t_{n+\frac{5}{3}} + \frac{\alpha^2 t_{n+\frac{5}{3}}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+\frac{5}{3}}^8}{8!}]$$
$$\eta_1 = [\alpha^3 + \alpha^4 t_n + \frac{\alpha^5 t_n^2}{2!} + \ldots + \frac{\alpha^8 t_n^5}{5!}]$$
$$\eta_2 = [\alpha^3 + \alpha^4 t_{n+1} + \frac{\alpha^5 t_{n+1}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+1}^5}{5!}]$$
$$\eta_3 = [\alpha^3 + \alpha^4 t_{n+2} + \frac{\alpha^5 t_{n+2}^2}{2!} + \ldots + \frac{\alpha^8 t_{n+2}^5}{5!}]$$
$$\psi = j(j-1)(j-2) \text{ as } j = 5, 6.$$

Solving (6) for real unknown parameters λ_j's and substituting back into (4), with some manipulation yields, a linear multistep method with continuous coefficients in the form:

$$y(t) = \sum_{j=0}^{k-1} \zeta_j(t) y_{n+j} + \left\{ \tau_1(t) y_{n+r} + \tau_2(t) y_{n+s} + \tau_3(t) y_{n+u} + \tau_4(t) y_{n+v} \right\} + h^3 \sum_{j=0}^{k} \beta_j(t) f_{n+j} \,. \qquad (7)$$

Taking $k = 2$, the coefficients $\zeta_j(t)$ and $\beta_j(t)$ are expressed as function of $v = \frac{t - t_{n+1}}{h}$ as follows:

$$\zeta_0(v) = \frac{913}{5530} v^2 - \frac{1215}{632} v^4 + \frac{12879}{3160} v^6 - \frac{729}{553} v^8$$

$$\zeta_1(v) = 1 - \frac{28213}{2212} v^2 + \frac{11907}{316} v^4 - \frac{11421}{316} v^6 + \frac{22577}{2212} v^8$$

$$\tau_1(v) = \frac{61}{1092}v - \frac{239437}{172536}v^2 + \frac{124497}{8216}v^4 - \frac{243}{52}v^5 - \frac{204039}{8216}v^6 + \frac{243}{182}v^7 + \frac{447849}{57512}v^8$$

$$\tau_2(v) = -\frac{440}{273}v + \frac{172033}{21567}v^2 - \frac{149769}{4108}v^4 + \frac{243}{26}v^5 + \frac{197721}{4108}v^6 - \frac{243}{91}v^7 - \frac{209709}{14378}v^8$$

$$\tau_3(v) = \frac{440}{273}v + \frac{271939}{43134}v^2 - \frac{139563}{8216}v^4 - \frac{243}{26}v^5 + \frac{56295}{8216}v^6 + \frac{243}{91}v^7 - \frac{17739}{14378}v^8$$

$$\tau_4(v) = -\frac{61}{1092}v - \frac{262273}{862680}v^2 + \frac{20817}{8216}v^4 + \frac{243}{52}v^5 + \frac{78813}{81080}v^6 - \frac{243}{182}v^7 - \frac{49815}{57512}v^8$$

$$\beta_0(v) = \frac{4}{331695}v + \frac{41359}{104815620}v^2 - \frac{77}{16432}v^4 - \frac{1}{780}v^5 + \frac{1327}{123240}v^6 + \frac{1}{364}v^7 - \frac{575}{115024}v^8$$

$$\beta_1(v) = -\frac{4778}{331695}v - \frac{423632}{26203905}v^2 + \frac{1}{6}v^3 + \frac{580}{3081}v^4 - \frac{67}{195}v^5 - \frac{6148}{15405}v^6 + \frac{17}{182}v^7 + \frac{928}{7189}v^8$$

$$\beta_2(v) = \frac{4}{331695}v + \frac{9769}{104815620}v^2 - \frac{46}{49296}v^4 - \frac{1}{780}v^5 + \frac{157}{123240}v^6 + \frac{1}{364}v^7 + \frac{127}{115024}v^8 \tag{8}$$

Evaluating (8) at $v = 1$ gives the discrete method

$$y_{n+2} = \frac{256}{39}y_{n+\frac{5}{3}} - \frac{395}{39}y_{n+\frac{4}{3}} + \frac{395}{39}y_{n+\frac{2}{3}} - \frac{256}{39}y_{n+\frac{1}{3}} + y_n + \frac{h^3}{9477}\left(28f_{n+2} - 1856f_{n+1} + 28f_n\right). \tag{9}$$

The first and second derivatives of (9) are:

$$hy'_{n+2} = \frac{100048}{35945}y_{n+\frac{5}{3}} - \frac{1452853}{28756}y_{n+\frac{4}{3}} - \frac{5480}{553}y_{n+1} + \frac{493522}{7189}y_{n+\frac{2}{3}} - \frac{307952}{7189}y_{n+\frac{1}{3}} + \frac{72421}{11060}y_n + \frac{h^3}{8734635}\left(222904f_{n+2} - 10653608f_{n+1} + 170254f_n\right). \tag{10}$$

$$h^2 y''_{n+2} = \frac{8220428}{107835} y_{n+\frac{5}{3}} - \frac{11174753}{86268} y_{n+\frac{4}{3}} - \frac{47276}{553} y_{n+1} + \frac{12093337}{43134} y_{n+\frac{2}{3}} - \frac{3607720}{21567} y_{n+\frac{1}{3}} + \frac{284317}{11060} y_n + \frac{h^3}{52407810}\left(9331249 f_{n+2} - 234051128 f_{n+1} + 4055719 f_n\right). \quad (11)$$

Applying the truncation error formula in [5], associated with equation (7) by the difference operator (10) to determine the order and error constant of the methods:

$$L[y(t):h] = \sum_{j=0}^{k} \left[\begin{array}{l} \alpha_j y(t_n + jh) + \{\tau_1 y(t_n + jhr) + \tau_2 y(t_n + jhs) + \tau_3 y(t_n + jhu) + \tau_4 y(t_n + jhv)\} \\ -h^3 \beta_j y'''(t_n + jh) \end{array} \right] \quad (12)$$

where $y(t)$ is assumed to be continuously differentiable of high order. Therefore, expanding (12) in Taylor's series and comparing the coefficient of h to give the expression

$$L[y(t):h] = c_0 y(t) + c_1 h y'(t) + c_2 h^2 y''(t) + \ldots + c_p h^p y^{(p)}(t) + \ldots + c_{p+3} h^{p+3} y^{(p+3)}(t) \quad (13)$$

The linear operator L and the associated methods are said to be of order p if $c_0 = c_1 = c_2 = \ldots = c_p = \ldots = c_{p+2} = 0$, $c_{p+3} \neq 0$. c_{p+3} is equal to the error constant. For the purpose of this work, expanding methods (8), (10) and (11) in Taylor's series and comparing the coefficient of h gives both methods of order $p = 6$ and error constant, $c_{p+3} = -3.633772 \times 10^{-6}$, $c_{p+3} = -1.1804754 \times 10^{-3}$ and $c_{p+3} = -1.029201 \times 10^{-4}$ respectively.

Equations (9)-(11) are of order six, symmetric, consistent, low error constants and capable of handling oscillatory problems.

3. Implementation of the Method

To implement the implicit linear 2-step 4-point discrete scheme (9) and its first and second derivatives (10) and (11) respectively, the following symmetric explicit schemes and their derivatives are also developed by the same procedure for the evaluation of y_{n+2}, y'_{n+2} and y''_{n+2}.

$$y_{n+2} = \frac{10392}{2629} y_{n+\frac{5}{3}} - \frac{15753}{2629} y_{n+\frac{4}{3}} + \frac{11312}{2629} y_{n+1} - \frac{3903}{2629} y_{n+\frac{1}{3}} - \frac{115}{2629} y_n + \frac{h^3}{354915}\left(6636 f_{n+\frac{5}{3}} - 4780 f_{n+1} - 56 f_n\right). \quad (14)$$

$$p = 6, \quad c_{p+3} = 3.107634 \times 10^{-7}$$

$$hy'_{n+2}=\frac{46152}{8365}y_{n+\frac{5}{3}}-\frac{70731}{4780}y_{n+\frac{4}{3}}+\frac{227992}{8365}y_{n+1}-\frac{264798}{8365}y_{n+\frac{2}{3}}+\frac{19296}{1195}y_{n+\frac{1}{3}}-\frac{16511}{6692}y_n+\frac{h^3}{376425}\left(60792f_{n+\frac{5}{3}}+133840f_{n+1}-2782f_n\right). \tag{15}$$

$$p=6,\quad c_{p+3}=9.0\times10^{-6}.$$

$$h^2y''_{n+2}=-\frac{29861988}{368060}y_{n+\frac{5}{3}}+\frac{44066883}{368060}y_{n+\frac{4}{3}}-\frac{63973128}{368060}y_{n+1}-\frac{154388262}{368060}y_{n+\frac{2}{3}}+\frac{89899548}{368060}y_{n+\frac{1}{3}}-\frac{13689309}{368060}y_n+h^3\left(\frac{9331249}{8281350}f_{n+\frac{5}{3}}+\frac{135647}{20790}f_{n+1}-\frac{1955008}{1774575}f_n\right). \tag{16}$$

$$p=6,\quad c_{p+3}=1.35\times10^{-4}.$$

The methods (14), (15) and (16) are of order $p=6$ and error constant, $c_{p+3}=3.107634\times10^{-7}$, $c_{p+3}=9.0\times10^{-6}$ and $c_{p+3}=1.35\times10^{-4}$ respectively.

Other explicit schemes were also generated to evaluate other starting values and Taylor's series was used to evaluate the values for $y_{n+i},\quad i=\frac{1}{3},\ \frac{2}{3},\ 1,$ as

$$y_{n+i}=y(t_n+ih)=y_n+ihy'_n+\frac{(ih)^2}{2!}y''_n+\frac{(ih)^3}{3!}f_n+\frac{(ih)^4}{4!}f'_n+\frac{(ih)^5}{5!}f''_n+\frac{(ih)^6}{6!}f''''_n.$$

$$y'_{n+i}=y'(t_n+ih)=y'_n+ihy''_n+\frac{(ih)^2}{2!}f_n+\frac{(ih)^3}{3!}f'_n+\frac{(ih)^4}{4!}f''_n+\frac{(ih)^5}{5!}f'''_n.$$

and

$$y''_{n+i}=y''(t_n+ih)=y''_n+ihf_n+\frac{(ih)^2}{2!}f'_n+\frac{(ih)^3}{3!}f''_n+\frac{(ih)^4}{4!}f'''_n. \tag{17}$$

4. Numerical Experiments

Three third order problems out of which one is linear and two are non-linear with exact solutions are solved with our method to test the effectiveness and its accuracy.

Problem 1.

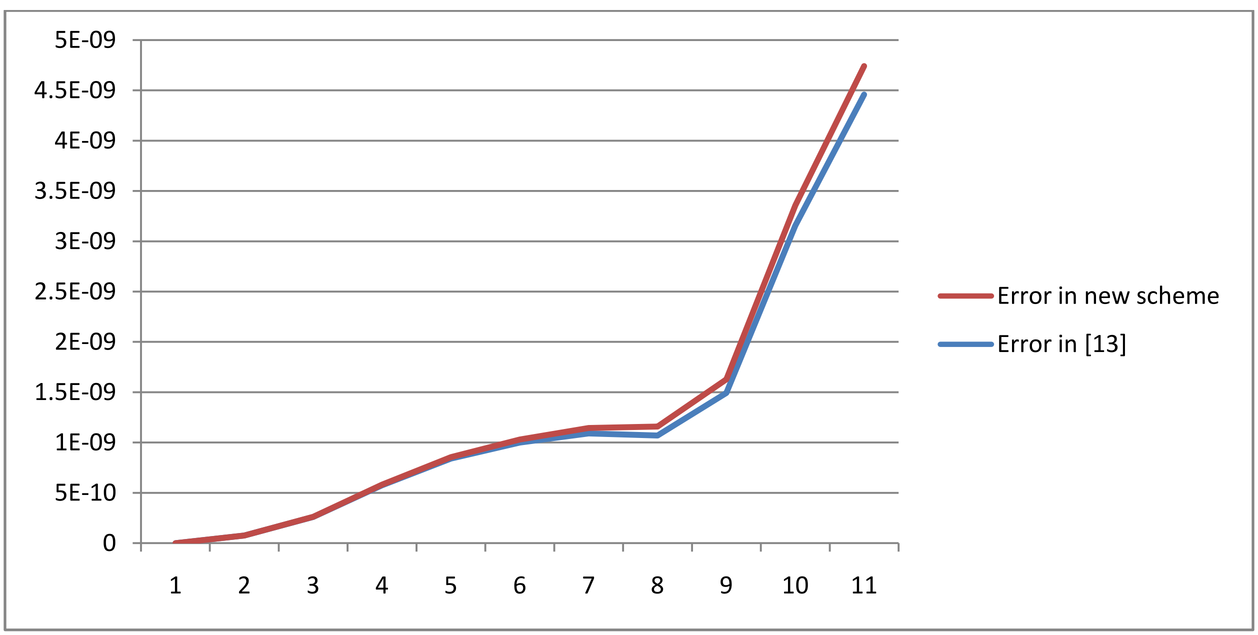
$$y''' = -e^t, \;\; y(0) = 1, \;\; y'(0) = -1, \;\; y''(0) = 3, h = 0.1$$

Theoretical solution: $y(t) = 2 + 2t^2 - e^t$.

Table 1: Shows the numerical solution of our methods of order 6 compared with the method of [13], of order 7.

t	y_{exact}	$y_{computed}$	Error in [13], $p = 7, k = 5.$	Error in new scheme, $p = 6, k = 2.$
0.1	0.9148290819243523	0.9148290819245347	7.56477e-11	1.82410e-13
0.2	0.8585972418398302	0.8585972418415010	2.60170e-10	1.67078e-12
0.3	0.8301411924239970	0.8301411924299984	5.76003e-10	6.00142e-12
0.4	0.8281753023587299	0.8281753023735897	8.41270e-10	1.48598e-11
0.5	0.8512787292998718	0.8512787293299923	1.00013e-09	3.01205e-11
0.6	0.8978811996094913	0.8978811996633331	1.09051e-09	5.38418e-11
0.7	0.9662472925295238	0.9662472926178395	1.07048e-09	8.83157e-11
0.8	1.0544590715075328	1.0544590716435931	1.49247e-09	1.36060e-10
0.9	1.1603968888430511	1.1603968890429206	3.15695e-09	1.99870e-10
1.0	1.2817181715409554	1.2817181718237693	4.45905e-09	2.82814e-10

Graph 1: Graph of error in new scheme against error in [13].



ISSN 2229-5518

Table 1 and Graph 1 show the maximum absolute error of our predictor-corrector method and that of [13] block method for Problem 1. It reveals that the new method performed creditably well than that of [13] of higher order.

Problem 2.

$$y''' = y'(2ty'' + y'),\ \ y(0) = 1,\ \ y'(0) = \frac{1}{2},\ \ y''(0) = 0,\ \ h = 0.01$$

Theoretical solution: $y(t) = 1 + \frac{1}{2}\ln\left[\frac{2+t}{2-t}\right].$

Table 2: Shows the numerical solution of our methods of order 6 compared with the method of [18], of order 6.

t	y_{exact}	$y_{computed}$	Error in [18] $p=6,\ k=3, h=0.01$	Error in new scheme, $p=6,\ k=2,$ $h=0.01$
0.21	1.105388447838499000	1.105388448780131800	8.037948e-11	9.416328e-10
0.31	1.156259497799360100	1.156259501147486100	6.043090e-10	3.348126e-09
0.41	1.207946365635211800	1.207946374195495700	2.581908e-09	8.560284e-09
0.51	1.260753316593162600	1.260753334825470000	8.158301e-09	1.823231e-08
0.61	1.315023237096001100	1.315023271700113600	2.141286e-08	3.460411e-08
0.71	1.371153208259014500	1.371153268976354400	4.969641e-08	6.071734e-08
0.81	1.429615588111108300	1.429615688840239200	1.620387e-07	1.007291e-07

Graph 2: Graph of error in new scheme against error in [18].

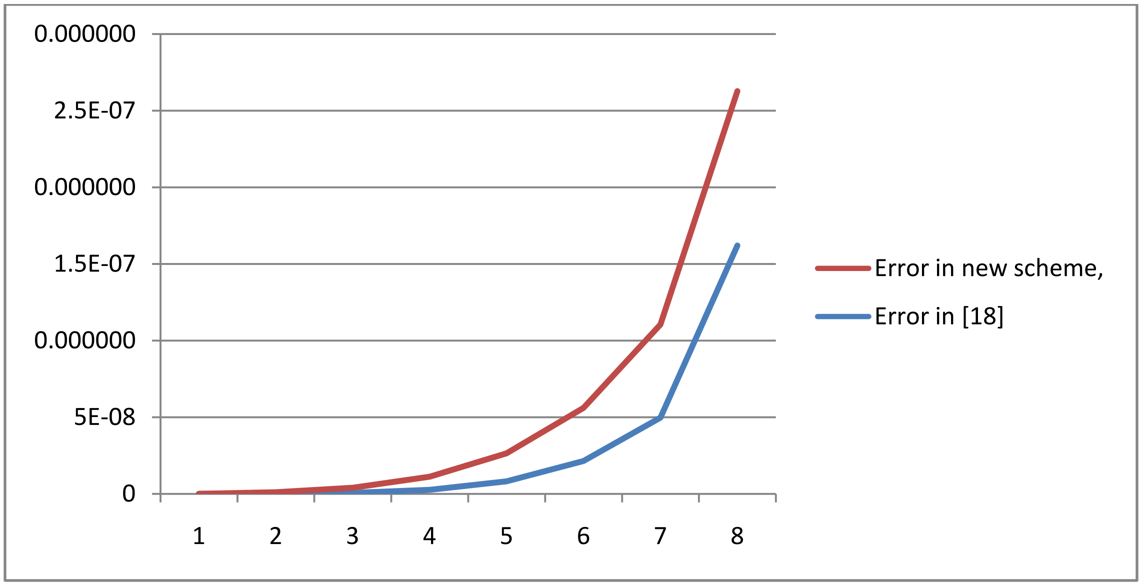


Table 2 and Graph 2 show the maximum absolute error of our predictor-corrector method and that of [19] block method for Problem 2. It reveals that the new method compared favorably with that of [19].

Problem 3.

$$y''' = 3\sin t, \;\; y(0) = 1, \;\; y'(0) = 0, \;\; y''(0) = -2, \;\; h = 0.1$$

Theoretical solution: $y(t) = 3\cos t + \frac{t^2}{2} - 2.$

Table 3: Numerical solution and errors for Problem 3

t	y_{exact}	$y_{computed}$	Error in [19] $p = 6, k = 2, h = 0.1$	Error in new scheme, $p = 6, k = 2, h = 0.1$
0.1	0.9900124958340770	0.9900124958340671	2.5934e-12	9.880985e-15
0.2	0.9601997335237251	0.9601997335235236	1.1857e-11	2.015055e-13
0.3	0.9110094673768181	0.9110094673756983	2.6224e-11	1.119771e-12
0.4	0.8431829820086554	0.8431829820049489	4.7034e-11	3.706480e-12
0.5	0.7577476856711178	0.7577476856618458	7.2700e-11	9.272028e-12
0.6	0.6560068447290348	0.6560068447095185	1.0437e-11	1.951628e-11
0.7	0.5395265618534650	0.5395265618169916	1.4049e-11	3.647349e-11
0.8	0.4101201280414957	0.4101201279789739	1.8197e-10	6.252177e-11
0.9	0.2698299048119925	0.2698299047116487	2.2736e-10	1.003438e-10
1.0	0.1209069176044184	0.1209069174515090	2.7729e-10	1.529094e-10

Graph 3: Graph of error in new scheme against error in [19].

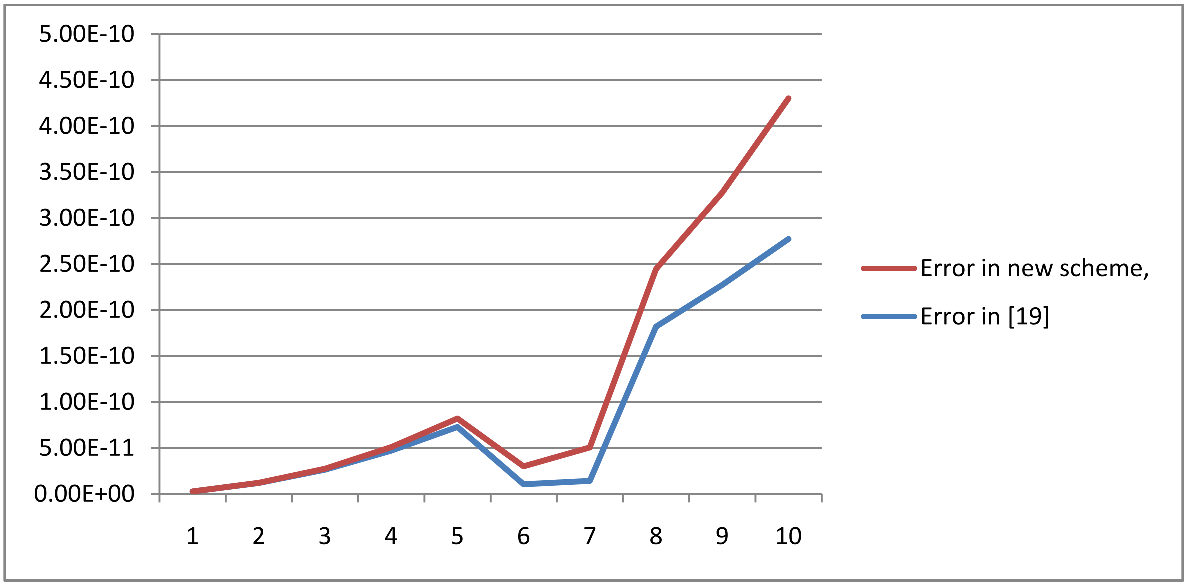


Table 3 show the maximum absolute error of our predictor-corrector method and that of [19] block method for Problem 3 and Graph 3 shows the graph of error in new scheme against error in [19]. It was revealed that the new method compared favorably with [19].

Problem 4.

$$y''' = -6(y)^4, \ \ y(1) = -1, \ \ y'(1) = -1, y''(1) = -2, \ \ h = \frac{1}{20}.$$

Theoretical solution: $y(t) = \dfrac{1}{t-2}$

Table 4: Numerical solution and errors for Problem 4

t	y_{exact}	$y_{computed}$	Error in new scheme, $p=6, \ k=2, h=\frac{1}{20}$	Time(s)
1.05	1.0526315789473684	1.0526315779293796	1.017989e-09	0.0291
1.10	1.1111111111111112	1.1111110853730750	2.573804e-08	0.0313
1.15	1.1764705882352944	1.1764704664689964	1.217663e-07	0.0316
1.20	1.2500000000000002	1.2499996375184190	3.624816e-07	0.0320
1.25	1.3333333333333337	1.3333324687923525	8.645410e-07	0.0323
1.30	1.4285714285714290	1.4285696093065769	1.819265e-06	0.0325
1.35	1.5384615384615392	1.5384579871365800	3.551325e-06	0.0328
1.40	1.6666666666666676	1.6666600337776776	6.632889e-06	0.0332
1.45	1.8181818181818195	1.8181697022691894	1.211591e-05	0.0335
1.50	2.0000000000000018	1.9999779695058428	2.203049e-05	0.0338

Table 4 shows the y-exact, the y-computed, the errors of the new method and the time(s) of iteration for Problem 4.

5. Conclusion

This paper has produced 2-step 4-point hybrid method for direct solution of higher order ordinary differential equations. Combination of power series and exponential function was used as basis function for the approximate solution to the given problem. The method is developed in such a way that the hybrid points are at the $y-function$ which enhanced the reduction of function evaluation. The new method developed is symmetric, consistent and convergent which handled oscillatory problems of type 4. The discrete scheme obtained was applied to test its usability and accuracy on linear and non-linear third order ODE problems. The results in Tables 1 and 4 revealed that the developed method performed more favorably than the existing methods implemented in block method.

References

[1] Jator, S. N. (2001). Improvements in Adams-Moulton Methods for the First Order Initial Value Problems, *Journal of the Tennessee Academy of Science, 76(2), 57-60.*

[2] Jator, S. N., Li, J. (2009). A Self Starting Linear Multistep Method for the General Second Order Initial Value Problems, *Intern. J. of Computer Mathematics 86(5): 817-836.*

[3] Kayode S. J., Adeyeye, O. (2011). A 3-step hybrid method for direct solution of second order initial value problems, *Aust. J. Basic Appl. Sci., 5(12), 2121-2126.*

[4] Areo E.A, Adeniyi, R.B. (2013). A Self-Starting Linear Multistep Method for Direct Solution of Initial Value Problems of Second Order Ordinary Differential Equations. *Intern. J. of Pure and Applied Mathematics,82(3): 345-364.*

[5] Awoyemi, D. O., Kayode, S.J., Adoghe, L.O. (2014). A Four-PointFully Implicit Method for Numerical Integration of Third-Order Ordinary Differential Equations, *Int. J. Physical Sciences, 9(1), 7-12*.

[6] Kuboye J. O., Omar, Z. (2015). Numerical Solution of Third order Ordinary Differential Equations Using a Seven-Step Block Method, *Intern. J. of Mathematical Analysis, 9(15), 743-754.*

[7] Yap, L.K., Ismail, F., Senu, N. (2014). An Accurate Block Hybrid Collocation Method for Third Order Ordinary Differential equations, *Journal of Applied Mathematics, ID 549597*, *http://dx.doi.org/10.1155/2014/549597*

[8] Majid, Z.A., Azimi, N.A., Suleiman, M. (2009). Solving second order ordinary differential Equations using two point four step direct implicit block method, *Euro. Journal of Scientific Research, 31(1), 29-36.*

[9] Akinfenwa, O. (2013). A. Ninth Order Block Piecewise Continuous Hybrid Integrators for Solving Second Order Ordinary Differential Equations, *Intern. J. of Diff. Equations and Appli., 12(1), 49-67.*

[10] Ogunware B. G, Omole E. O., and Olanegan O. O (2015): Hybrid and Non-Hybrid Implicit Schemes for solving Third Order ODEs Using Block Method as Predictors. *Journal of Mathematical Theory and Modelling (iiste) 5(3): 10 -25*

[11] Awoyemi, D. O., Idowu, O. (2005). A Class of Hybrid Collocation Methods for Third Order Ordinary Differential Equations, *Int. J. Computer Math., 82(10), 1287-1293.*

[12] Olabode, B.T. and Yusuf, Y. (2009). A New Block Method for Special Third Order Ordinary Differential Equations. *Journal of Mathematics and Statistics, 53: 167-170.*

[13] Olabode, B.T. (2009). An Accurate Scheme by Block Method for Third Order Ordinary Differential Equations, *Pacific Journal of Science and Technology,10(1), (2009), 136-142.*

[14] Olabode, B.T. (2013). Block Multistep Method for the Direct Solution of Third Order of Ordinary Differential Equations, *FUTA Journal of Research in Sciences, (2), 194-200.*

[15] Okunuga, S.A. Ehijie, J. (2009). New Derivation of Continuous Multistep Methods using Power Series as Basis Function, *J. Modern Math. Stat. 3(2).*

[16] Odekunle M.R, Adesanya A.O, and Sunday J (2012): A new block integrator for the solution of initial value problems of first order ordinary differential equations. *Int. J. Pure Applied Science Technology. 11(1) 92 – 100.*

[17] F.O. Obarhua and S.J. Kayode, “Symmetric Hybrid Linear Multistep Method for General Third Order Differential Equations,” *Open Access Library Journal*, ***3**:e2583,* *http://dx.doi.org/10.4236/oalib.1102583* 2016.

[18] A.O. Adesanya, Block Methods for Direct Solution of General Higher order Initial Value Problems of Ordinary Differential Equations, *PhD Thesis, Federal University of Technology, Akure.* 2011. *(Unpublished).*

[19] A.O. Adesanya, D.M. Udoh and A.M. Ajileye, “A New Hybrid Block Method for the Solution of General Third order Initial Value problems of Ordinary Differential Equations,” *Inter. J. of Pure and Appl. Math. 86(2): 365-375,* 2013.